Algebraic Number Theory
Exercise Sheet 8

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Exercise 1. Recall that a subset $X \subseteq \mathbb{R}^n$ is said to be *discrete*, if $X \cap B$ is finite for every compact $B \subseteq \mathbb{R}^n$.
(1) Let $X \subseteq \mathbb{R}^n$ be a discrete subset. Show that the induced topology on X is discrete.
(2) Let X be an additive subgroup of $\mathbb{R}^n$, such that the induced topology on X is discrete. Show that $X \subseteq \mathbb{R}^n$ is a discrete subset.

Exercise 2. Let d be a square-free positive integer, $d \not\equiv 1 \bmod 4$. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field.
(1) Find r_2 and d_K. Compute $2^{-r_2}|d_K|^{1/2}$.
(2) Let L be the lattice in $\mathbb{R}^2$ generated by $(1,1)$ and $(\sqrt{d}, -\sqrt{d})$. Compute explicitely the volume of L.
(3) Compare results in (1) and (2) and give an explanation.

Exercise 3. Let K be a number field and let A be the ring of integers of K. Let I, J be fractional ideals in $\mathcal{F}(A)$. Show that $N(IJ) = N(I)N(J)$.
Hint: Reduce the problem to the case where I and J are ideals in A and I is prime.

Exercise 4. Let K be a number field and let A be the ring of integers of K. Let ρ be a prime ideal in $\mathcal{F}(A)$. Show that $N(\rho)$ is a power of a prime number in $\mathbb{Z}$.